BOUNDING THE ČEBYŠEV FUNCTIONAL FOR A PAIR OF SEQUENCES IN INNER PRODUCT SPACES

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ABSTRACT. Some new bounds for the Čebyšev functional of a pair of vectors in inner product spaces are pointed out. Reverses for the celebrated Jensen's inequality for convex functions defined on inner product spaces are given as well.

1. INTRODUCTION

Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product over the real or complex number field $\mathbb{K}$. For $\mathbf{p}=(p_1,\dots,p_n)\in\mathbb{R}^n$ and $\mathbf{x}=(x_1,\dots,x_n)$, $\mathbf{y}=(y_1,\dots,y_n)\in H^n$, define the *Čebyšev functional*

$$T_n(\mathbf{p};\mathbf{x},\mathbf{y}):=P_n\sum_{i=1}^n p_i\langle x_i,y_i\rangle-\left\langle\sum_{i=1}^n p_ix_i,\sum_{i=1}^n p_iy_i\right\rangle, \tag{1.1}$$

where $P_n:=\sum_{i=1}^n p_i$.

The following Grüss type inequality has been obtained in [1].

Theorem 1. *Let H, $\mathbf{x},\mathbf{y}$ be as above and $p_i\geq 0$ $(i\in\{1,\dots,n\})$ with $\sum_{i=1}^n p_i=1$, i.e., $\mathbf{p}$ is a probability sequence. If $x,X,y,Y\in H$ are such that*

$$\operatorname{Re}\langle X-x_i,x_i-x\rangle\geq 0,\quad \operatorname{Re}\langle Y-y_i,y_i-y\rangle\geq 0 \tag{1.2}$$

for each $i\in\{1,\dots,n\}$, or equivalently, (see [2])

$$\left\|x_i-\frac{x+X}{2}\right\|\leq\frac{1}{2}\|X-x\|,\qquad\left\|y_i-\frac{y+Y}{2}\right\|\leq\frac{1}{2}\|Y-y\| \tag{1.3}$$

for each $i\in\{1,\dots,n\}$, then we have the inequality

$$|T_n(\mathbf{p};\mathbf{x},\mathbf{y})|\leq\frac{1}{4}\|X-x\|\|Y-y\|. \tag{1.4}$$

The constant $\frac{1}{4}$ is best possible in the sense that it cannot be replaced by a smaller constant.

In [3], the following Grüss type inequality for the forward difference of vectors was established.

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Theorem 2. *Let* $\mathbf{x}=(x_1,\dots,x_n)$, $\mathbf{y}=(y_1,\dots,y_n)\in H^n$ *and* $\mathbf{p}\in\mathbb{R}_+^n$ *be a probability sequence. Then one has the inequality:*

$$|T_n(\mathbf{p};\mathbf{x},\mathbf{y})| \leq \begin{cases} \left[\sum_{i=1}^n i^2 p_i - \left(\sum_{i=1}^n i p_i\right)^2\right] \max_{1\leq k\leq n-1}\|\Delta x_k\| \max_{1\leq k\leq n-1}\|\Delta y_k\|; \\ \sum_{1\leq j<i\leq n} p_i p_j (i-j)\left(\sum_{k=1}^{n-1}\|\Delta x_k\|^p\right)^{\frac{1}{p}}\left(\sum_{k=1}^{n-1}\|\Delta y_k\|^q\right)^{\frac{1}{q}} \\ \qquad\qquad \text{if } \ p>1,\ \frac{1}{p}+\frac{1}{q}+1 \\ \frac{1}{2}\left[\sum_{i=1}^n p_i(1-p_i)\right]\sum_{k=1}^{n-1}\|\Delta x_k\|\sum_{k=1}^{n-1}\|\Delta y_k\|. \end{cases} \tag{1.5}$$

The constants $1,1$ *and* $\frac{1}{2}$ *in the right hand side of inequality (1.5) are best in the sense that they cannot be replaced by smaller constants.*

Another result is incorporated in the following theorem (see [2]).

Theorem 3. *Let* $\mathbf{x},\mathbf{y}$ *and* $\mathbf{p}$ *be as in Theorem 2. If there exist* $x,X\in H$ *such that*

$$\operatorname{Re}\langle X-x_i, x_i-x\rangle \geq 0 \quad \text{for each} \ \ i\in\{1,\dots,n\}, \tag{1.6}$$

or, equivalently,

$$\left\|x_i-\frac{x+X}{2}\right\| \leq \frac{1}{2}\|X-x\| \quad \text{for each} \ \ i\in\{1,\dots,n\}, \tag{1.7}$$

then one has the inequality

$$\begin{aligned} |T_n(\mathbf{p};\mathbf{x},\mathbf{y})| &\leq \frac{1}{2}\|X-x\|\sum_{i=1}^n p_i\left\|y_i-\sum_{j=1}^n p_j y_j\right\| \\ &\leq \frac{1}{2}\|X-x\|\left[\sum_{i=1}^n p_i\|y_i\|^2-\left\|\sum_{i=1}^n p_i y_i\right\|^2\right]^{\frac{1}{2}}. \end{aligned} \tag{1.8}$$

The constant $\frac{1}{2}$ *is best possible in the first and second inequalities in the sense that it cannot be replaced by a smaller constant.*

Remark 1. *If* $\mathbf{x}$ *and* $\mathbf{y}$ *satisfy the assumptions of Theorem 1, then we have the following sequence of inequalities improving the Grüss inequality (1.4):*

$$\begin{aligned} |T_n(\mathbf{p};\mathbf{x},\mathbf{y})| &\leq \frac{1}{2}\|X-x\|\sum_{i=1}^n p_i\left\|y_i-\sum_{j=1}^n p_j y_j\right\| \\ &\leq \frac{1}{2}\|X-x\|\left(\sum_{i=1}^n p_i\|y_i\|^2-\left\|\sum_{i=1}^n p_i y_i\right\|^2\right)^{\frac{1}{2}} \\ &\leq \frac{1}{4}\|X-x\|\,\|Y-y\|. \end{aligned} \tag{1.9}$$

Now, if we consider the Čebyšev functional for the uniform probability distribution $u = \left(\frac{1}{n}, \dots, \frac{1}{n}\right)$,

$$T_n(\mathbf{x}, \mathbf{y}) := \frac{1}{n} \sum_{i=1}^{n} \langle x_i, y_i \rangle - \left\langle \frac{1}{n} \sum_{i=1}^{n} x_i, \frac{1}{n} \sum_{i=1}^{n} y_i \right\rangle,$$

then, with the assumptions of Theorem 1, we have

$$|T_n(\mathbf{x}, \mathbf{y})| \leq \frac{1}{4} \|X - x\| \|Y - y\|. \tag{1.10}$$

Theorem 2 will provide the following inequalities

$$|T_n(\mathbf{x}, \mathbf{y})| \tag{1.11}$$
$$\leq \begin{cases} \frac{1}{12}\left(n^2 - 1\right) \max\limits_{1 \leq k \leq n-1} \|\Delta x_k\| \max\limits_{1 \leq k \leq n-1} \|\Delta y_k\|; \\[2ex] \frac{1}{6}\left(n - \frac{1}{n}\right) \left(\sum\limits_{k=1}^{n-1} \|\Delta x_k\|^p\right)^{\frac{1}{p}} \left(\sum\limits_{k=1}^{n-1} \|\Delta y_k\|^q\right)^{\frac{1}{q}} & \text{if } \ p > 1, \ \frac{1}{p} + \frac{1}{q} + 1; \\[2ex] \frac{1}{2}\left(1 - \frac{1}{n}\right) \sum\limits_{k=1}^{n-1} \|\Delta x_k\| \sum\limits_{k=1}^{n-1} \|\Delta y_k\|. \end{cases}$$

Here the constants $\frac{1}{12}$, $\frac{1}{6}$ and $\frac{1}{2}$ are best possible in the above sense.

Finally, from (1.9), we have

$$\begin{aligned} |T_n(\mathbf{x}, \mathbf{y})| &\leq \frac{1}{2n} \|X - x\| \sum_{i=1}^{n} \left\| y_i - \frac{1}{n} \sum_{j=1}^{n} y_j \right\| \\ &\leq \frac{1}{2} \|X - x\| \left(\frac{1}{n} \sum_{i=1}^{n} \|y_i\|^2 - \left\| \frac{1}{n} \sum_{i=1}^{n} y_i \right\|^2 \right)^{\frac{1}{2}} \\ &\leq \frac{1}{4} \|X - x\| \|Y - y\|. \end{aligned} \tag{1.12}$$

It is the main aim of this paper to point out other bounds for the Čebyšev functionals $T_n(\mathbf{p}; \mathbf{x}, \mathbf{y})$ and $T_n(\mathbf{x}, \mathbf{y})$. Applications for Jensen's inequality for convex functions defined on inner product spaces are given as well.

2. Identities for Inner Products

For $\mathbf{p} = (p_1, \dots, p_n) \in \mathbb{R}^n$ and $\mathbf{a} = (a_1, \dots, a_n) \in H^n$ we define

$$P_i := \sum_{k=1}^{i} p_k, \qquad \bar{P}_i = P_n - P_i, \qquad i \in \{1, \dots, n-1\}$$

and the vectors

$$A_i(\mathbf{p}) = \sum_{k=1}^{i} p_k a_k, \qquad \bar{A}_i(\mathbf{p}) = A_n(\mathbf{p}) - A_i(\mathbf{p})$$

for $i \in \{1, \dots, n-1\}$.

The following result holds.

Theorem 4. *Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{K}$, $\mathbf{p}=(p_1,\dots,p_n)\in\mathbb{R}^n$ and $\mathbf{a}=(a_1,\dots,a_n)\,,\mathbf{b}=(b_1,\dots,b_n)\in H^n$. Then we have the identities*

$$
\begin{aligned}
T_n(\mathbf{p};\mathbf{a},\mathbf{b}) &= \sum_{i=1}^{n-1}\langle P_iA_n(\mathbf{p})-P_nA_i(\mathbf{p}),\Delta b_i\rangle \\
&= P_n\sum_{i=1}^{n-1}P_i\left\langle \frac{1}{P_n}A_n(\mathbf{p})-\frac{1}{P_i}A_i(\mathbf{p}),\Delta b_i\right\rangle \\
&\text{(if } P_i\neq 0,\ i\in\{1,\dots,n\}\text{)} \\
&= \sum_{i=1}^{n-1}P_i\bar{P}_i\left\langle \frac{1}{\bar{P}_i}\bar{A}_i(\mathbf{p})-\frac{1}{P_i}A_i(\mathbf{p}),\Delta b_i\right\rangle \\
&\text{(if } P_i,\bar{P}_i\neq 0,\ i\in\{1,\dots,n-1\}\text{),}
\end{aligned}
\tag{2.1}
$$

where $\Delta x_i=x_{i+1}-x_i$ $(i\in\{1,\dots,n-1\})$ is the forward difference.

Proof. We use the following summation by parts formula for vectors in inner product spaces

$$
\sum_{l=p}^{q-1}\langle d_l,\Delta v_l\rangle=\langle d_l,v_l\rangle\Big|_p^q-\sum_{l=p}^{q-1}\langle v_{l+1},\Delta d_l\rangle\,,
\tag{2.2}
$$

where d_l, v_l are vectors in H, $l=p,\dots,q$ $(q>p;\ p,q$ are natural numbers).

If we choose in (2.2), $p=1$, $q=n$, $d_i=P_iA_n(\mathbf{p})-P_nA_i(\mathbf{p})$ and $v_i=b_i$ $(i\in\{1,\dots,n-1\})$, then we get

$$
\begin{aligned}
&\sum_{i=1}^{n-1}\langle P_iA_n(\mathbf{p})-P_nA_i(\mathbf{p}),\Delta b_i\rangle \\
=\ & \langle P_iA_n(\mathbf{p})-P_nA_i(\mathbf{p}),b_i\rangle\Big|_1^n-\sum_{i=1}^{n-1}\langle\Delta(P_iA_n(\mathbf{p})-P_nA_i(\mathbf{p})),b_{i+1}\rangle \\
=\ & \langle P_nA_n(\mathbf{p})-P_nA_n(\mathbf{p}),b_n\rangle-\langle P_1A_n(\mathbf{p})-P_nA_1(\mathbf{p}),b_1\rangle \\
& -\sum_{i=1}^{n-1}\langle P_{i+1}A_n(\mathbf{p})-P_nA_{i+1}(\mathbf{p})-P_iA_n(\mathbf{p})+P_nA_i(\mathbf{p}),b_{i+1}\rangle \\
=\ & P_np_1\langle a_1,x_1\rangle-p_1\langle A_n(\mathbf{p}),b_1\rangle-\left\langle A_n(\mathbf{p}),\sum_{i=1}^{n-1}p_{i+1}b_{i+1}\right\rangle \\
& +P_n\sum_{i=1}^{n-1}p_{i+1}\langle a_{i+1},b_{i+1}\rangle \\
=\ & P_n\sum_{i=1}^{n}p_i\langle a_i,b_i\rangle-\left\langle\sum_{i=1}^{n}p_ia_i,\sum_{i=1}^{n}p_ib_i\right\rangle \\
=\ & T_n(\mathbf{p};\mathbf{a},\mathbf{b})\,,
\end{aligned}
$$

proving the first identity in (2.1).

The second and third identities are obvious and we omit the details. ∎

The following lemma is of interest in itself.

Lemma 1. *Let* $\mathbf{p} = (p_1, \ldots, p_n) \in \mathbb{R}^n$ *and* $\mathbf{a} = (a_1, \ldots, a_n) \in H$. *Then we have the equality*

$$P_i A_n(\mathbf{p}) - P_n A_i(\mathbf{p}) = \sum_{j=1}^{n-1} P_{\min\{i,j\}} \bar{P}_{\max\{i,j\}} \Delta a_j \tag{2.3}$$

for each $i \in \{1, \ldots, n-1\}$.

Proof. Define, for $i \in \{1, \ldots, n-1\}$, the vector

$$K(i) := \sum_{j=1}^{n-1} P_{\min\{i,j\}} \bar{P}_{\max\{i,j\}} \cdot \Delta a_j.$$

We have

$$\begin{aligned} K(i) &= \sum_{j=1}^{i} P_{\min\{i,j\}} \bar{P}_{\max\{i,j\}} \cdot \Delta a_j + \sum_{j=i+1}^{n-1} P_{\min\{i,j\}} \bar{P}_{\max\{i,j\}} \cdot \Delta a_j \\ &= \sum_{j=1}^{i} P_j \bar{P}_i \cdot \Delta a_j + \sum_{j=i+1}^{n-1} P_i \bar{P}_j \cdot \Delta a_j \\ &= \bar{P}_i \sum_{j=1}^{i} P_j \cdot \Delta a_j + P_i \sum_{j=i+1}^{n-1} \bar{P}_j \cdot \Delta a_j. \end{aligned} \tag{2.4}$$

Using the summation by parts formula, we have

$$\begin{aligned} \sum_{j=1}^{i} P_j \cdot \Delta a_j &= P_j a_j \Big|_1^{i+1} - \sum_{j=1}^{i} (P_{j+1} - P_j) a_{j+1} \\ &= P_{i+1} a_{i+1} - p_1 a_1 - \sum_{j=1}^{i} p_{j+1} a_{j+1} \\ &= P_{i+1} a_{i+1} - \sum_{j=1}^{i+1} p_j a_j \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} \sum_{j=i+1}^{n-1} \bar{P}_j \cdot \Delta a_j &= \bar{P}_j a_j \Big|_{i+1}^{n} - \sum_{j=i+1}^{n-1} \left(\bar{P}_{j+1} - \bar{P}_j\right) a_{j+1} \\ &= \bar{P}_n a_n - \bar{P}_{i+1} a_{i+1} - \sum_{j=i+1}^{n-1} (P_n - P_{j+1} - P_n + P_j) a_{j+1} \\ &= -\bar{P}_{i+1} a_{i+1} + \sum_{j=i+1}^{n-1} p_{j+1} a_{j+1}. \end{aligned} \tag{2.6}$$

Using (2.5) and (2.6), we have

$$
\begin{aligned}
K(i) &= \bar{P}_i \left(P_{i+1} a_{i+1} - \sum_{j=1}^{i+1} p_j a_j \right) + P_i \left(\sum_{j=i+1}^{n-1} p_{j+1} a_{j+1} - \bar{P}_{i+1} a_{i+1} \right) \\
&= \bar{P}_i P_{i+1} a_{i+1} - \bar{P}_i \bar{P}_{i+1} a_{i+1} - \bar{P}_i \sum_{j=1}^{i+1} p_j a_j + P_i \sum_{j=i+1}^{n-1} p_{j+1} a_{j+1} \\
&= \left[(P_n - P_i) P_{i+1} - P_i (P_n - P_{i+1}) \right] a_{i+1} + P_i \sum_{j=i+1}^{n-1} p_{j+1} a_{j+1} - \bar{P}_i \sum_{j=1}^{i+1} p_j a_j \\
&= P_n p_{i+1} a_{i+1} + P_i \sum_{j=i+1}^{n-1} p_{j+1} a_{j+1} - \bar{P}_i \sum_{j=1}^{i+1} p_j a_j \\
&= \left(P_i + \bar{P}_i \right) p_{i+1} a_{i+1} + P_i \sum_{j=i+1}^{n-1} p_{j+1} a_{j+1} - \bar{P}_i \sum_{j=1}^{i+1} p_j a_j \\
&= P_i \sum_{j=i+1}^{n-1} p_j a_j - \bar{P}_i \sum_{j=1}^{i} p_j a_j \\
&= P_i \bar{A}_i (\mathbf{p}) - \bar{P}_i A_i (\mathbf{p}) \\
&= P_i A_n (\mathbf{p}) - P_n A_i (\mathbf{p}),
\end{aligned}
$$

and the identity is proved. ∎

We are able now to state and prove the second identity for the Čebyšev functional.

Theorem 5. *With the assumptions of Theorem 4, we have the identity*

$$
T_n (\mathbf{p}; \mathbf{a}, \mathbf{b}) = \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} P_{\min\{i,j\}} \bar{P}_{\max\{i,j\}} \cdot \langle \Delta a_j, \Delta b_i \rangle . \tag{2.7}
$$

Proof. Follows by Theorem 4 and Lemma 1 and we omit the details. ∎

3. New Inequalities

The following result holds.

Theorem 6. *Let $(H; \langle \cdot, \cdot \rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$; $\mathbf{p} = (p_1, \dots, p_n) \in \mathbb{R}^n$ and $\mathbf{a} = (a_1, \dots, a_n), \mathbf{b} = (b_1, \dots, b_n) \in H^n$. Then we have the inequalities*

$$
|T_n (\mathbf{p}; \mathbf{a}, \mathbf{b})| \leq \begin{cases}
\max\limits_{1 \leq i \leq n-1} \| P_i A_n (\mathbf{p}) - P_n A_i (\mathbf{p}) \| \sum\limits_{j=1}^{n-1} \| \Delta b_j \| ; \\[2ex]
\left(\sum\limits_{i=1}^{n-1} \| P_i A_n (\mathbf{p}) - P_n A_i (\mathbf{p}) \|^q \right)^{\frac{1}{q}} \left(\sum\limits_{j=1}^{n-1} \| \Delta b_j \|^p \right)^{\frac{1}{p}} \\
\qquad \qquad \text{for } \; p > 1, \; \frac{1}{p} + \frac{1}{q} = 1; \\[2ex]
\sum\limits_{i=1}^{n-1} \| P_i A_n (\mathbf{p}) - P_n A_i (\mathbf{p}) \| \cdot \max\limits_{1 \leq j \leq n-1} \| \Delta b_j \| .
\end{cases} \tag{3.1}
$$

All the inequalities in (3.1) are sharp in the sense that the constants 1 *cannot be replaced by smaller constants.*

Proof. Using the first identity in (2.1) and Schwarz's inequality in H, i.e., $|\langle u, v\rangle| \leq \|u\| \|v\|$, $u, v \in H$, we have successively:

$$\begin{aligned}|T_n(\mathbf{p}; \mathbf{a}, \mathbf{b})| &\leq \sum_{i=1}^{n-1} |\langle P_i A_n(\mathbf{p}) - P_n A_i(\mathbf{p}), \Delta b_i\rangle| \\ &\leq \sum_{i=1}^{n-1} \|P_i A_n(\mathbf{p}) - P_n A_i(\mathbf{p})\| \|\Delta b_i\|.\end{aligned}$$

Using Hölder's inequality, we deduce the desired result (3.1).

Let us prove, for instance, that the constant 1 in the second inequality is best possible.

Assume, for $c > 0$, we have that

$$|T_n(\mathbf{p}; \mathbf{a}, \mathbf{b})| \leq c \left(\sum_{i=1}^{n-1} \|P_i A_n(\mathbf{p}) - P_n A_i(\mathbf{p})\|^q\right)^{\frac{1}{q}} \left(\sum_{j=1}^{n-1} \|\Delta b_j\|^p\right)^{\frac{1}{p}} \tag{3.2}$$

for $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $n \geq 2$.

If we choose $n = 2$, then we get

$$T_2(\mathbf{p}; \mathbf{a}, \mathbf{b}) = p_1 p_2 \langle a_2 - a_1, b_2 - b_1\rangle.$$

Also, for $n = 2$,

$$\left(\sum_{i=1}^{n-1} \|P_i A_n(\mathbf{p}) - P_n A_i(\mathbf{p})\|^q\right)^{\frac{1}{q}} = |p_1 p_2| \|a_2 - a_1\|$$

and

$$\left(\sum_{j=1}^{n-1} \|\Delta b_j\|^p\right)^{\frac{1}{p}} = \|b_2 - b_1\|,$$

and then, from (3.2), for $n = 2$, we deduce

$$|p_1 p_2| |\langle a_2 - a_1, b_2 - b_1\rangle| \leq c |p_1 p_2| \|a_2 - a_1\| \|b_2 - b_1\|. \tag{3.3}$$

If in (3.3) we choose $a_2 = b_2$, $a_2 = b_1$ and $b_2 \neq b_1$, $p_1, p_2 \neq 0$, we deduce $c \geq 1$, proving that 1 is the best possible constant in that inequality. ∎

The following corollary for the uniform distribution of the probability $\mathbf{p}$ holds.

Corollary 1. *With the assumptions of Theorem 6 for* $\mathbf{a}$ *and* $\mathbf{b}$, *we have the inequalities*

$$0 \leq |T_n(\mathbf{a},\mathbf{b})| \leq \frac{1}{n^2} \times \begin{cases} \max\limits_{1\leq i\leq n-1} \left\| i\sum\limits_{k=1}^{n} a_k - n\sum\limits_{k=1}^{i} a_k \right\| \sum\limits_{j=1}^{n-1} \|\Delta b_j\|; \\ \left(\sum\limits_{i=1}^{n-1} \left\| i\sum\limits_{k=1}^{n} a_k - n\sum\limits_{k=1}^{i} a_k \right\|^q\right)^{\frac{1}{q}} \left(\sum\limits_{j=1}^{n-1} \|\Delta b_j\|^p\right)^{\frac{1}{p}} \\ \qquad \textit{for } \ p>1, \ \frac{1}{p}+\frac{1}{q}=1; \\ \sum\limits_{i=1}^{n-1} \left\| i\sum\limits_{k=1}^{n} a_k - n\sum\limits_{k=1}^{i} a_k \right\| \cdot \max\limits_{1\leq j\leq n-1} \|\Delta b_j\|. \end{cases} \tag{3.4}$$

The following result may be stated as well.

Theorem 7. *With the assumptions of Theorem 6 and if* $P_i \neq 0$ $(i=1,\dots,n)$, *then we have the inequalities*

$$\begin{aligned} &|T_n(\mathbf{p};\mathbf{a},\mathbf{b})| \\ &\leq |P_n| \times \begin{cases} \max\limits_{1\leq i\leq n-1} \left\| \frac{1}{P_n}A_n(\mathbf{p}) - \frac{1}{P_i}A_i(\mathbf{p}) \right\| \sum\limits_{i=1}^{n-1} |P_i| \|\Delta b_i\|; \\ \left(\sum\limits_{i=1}^{n-1} |P_i| \left\| \frac{1}{P_n}A_n(\mathbf{p}) - \frac{1}{P_i}A_i(\mathbf{p}) \right\|^q\right)^{\frac{1}{q}} \left(\sum\limits_{i=1}^{n-1} |P_i| \|\Delta b_i\|^p\right)^{\frac{1}{p}} \\ \qquad \textit{for } \ p>1, \ \frac{1}{p}+\frac{1}{q}=1; \\ \sum\limits_{i=1}^{n-1} |P_i| \left\| \frac{1}{P_n}A_n(\mathbf{p}) - \frac{1}{P_i}A_i(\mathbf{p}) \right\| \cdot \max\limits_{1\leq i\leq n-1} \|\Delta b_i\|. \end{cases} \end{aligned} \tag{3.5}$$

All the inequalities in (3.5) are sharp in the sense that the constant 1 cannot be replaced by a smaller constant.

Proof. Using the second equality in (2.1) and Schwarz's inequality, we have

$$\begin{aligned} |T_n(\mathbf{p};\mathbf{a},\mathbf{b})| &\leq |P_n| \sum_{i=1}^{n-1} |P_i| \left| \left\langle \frac{1}{P_n}A_n(\mathbf{p}) - \frac{1}{P_i}A_i(\mathbf{p}), \Delta b_i \right\rangle \right| \\ &\leq |P_n| \sum_{i=1}^{n-1} |P_i| \left\| \frac{1}{P_n}A_n(\mathbf{p}) - \frac{1}{P_i}A_i(\mathbf{p}) \right\| \|\Delta b_i\|. \end{aligned}$$

Using Hölder's weighted inequality, we deduce (3.5).

The sharpness of the constant may be proven in a similar manner to the one in Theorem 6. We omit the details. ∎

The following corollary containing the unweighted inequalities holds.

Corollary 2. *With the above assumptions for* $\mathbf{a}$ *and* $\mathbf{b}$, *one has*

$$(3.6)\qquad |T_n(\mathbf{a},\mathbf{b})| \le \frac{1}{n}\times\begin{cases} \max\limits_{1\le i\le n-1}\left\|\dfrac{1}{n}\sum\limits_{k=1}^{n}a_k-\dfrac{1}{i}\sum\limits_{k=1}^{i}a_k\right\|\sum\limits_{i=1}^{n-1}i\left\|\Delta b_i\right\|; \\[2ex] \left(\sum\limits_{i=1}^{n-1}i\left\|\dfrac{1}{n}\sum\limits_{k=1}^{n}a_k-\dfrac{1}{i}\sum\limits_{k=1}^{i}a_k\right\|^q\right)^{\frac{1}{q}}\left(\sum\limits_{i=1}^{n-1}i\left\|\Delta b_i\right\|^p\right)^{\frac{1}{p}} \\ \qquad\qquad\qquad\qquad\qquad \text{for } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\[2ex] \sum\limits_{i=1}^{n-1}i\left\|\dfrac{1}{n}\sum\limits_{k=1}^{n}a_k-\dfrac{1}{i}\sum\limits_{k=1}^{i}a_k\right\|\cdot\max\limits_{1\le i\le n-1}\left\|\Delta b_i\right\|. \end{cases}$$

The inequalities (3.6) are sharp in the sense mentioned above.

Another type of inequality may be stated if ones used the third identity in (2.1) and Hölder's weighted inequality with the weights: $|P_i|\left|\bar{P}_i\right|$, $i\in\{1,\dots,n-1\}$.

Theorem 8. *With the assumptions in Theorem 6 and if* $P_i, \bar{P}_i \ne 0$, $i\in\{1,\dots,n-1\}$, *then we have the inequalities*

$$(3.7)\qquad |T_n(\mathbf{p};\mathbf{a},\mathbf{b})|$$
$$\le |P_n|\times\begin{cases} \max\limits_{1\le i\le n-1}\left\|\dfrac{1}{\bar{P}_i}\bar{A}_i(\mathbf{p})-\dfrac{1}{P_i}A_i(\mathbf{p})\right\|\sum\limits_{i=1}^{n-1}|P_i|\left|\bar{P}_i\right|\left\|\Delta b_i\right\|; \\[2ex] \left(\sum\limits_{i=1}^{n-1}|P_i|\left|\bar{P}_i\right|\left\|\dfrac{1}{\bar{P}_i}\bar{A}_i(\mathbf{p})-\dfrac{1}{P_i}A_i(\mathbf{p})\right\|^q\right)^{\frac{1}{q}}\left(\sum\limits_{i=1}^{n-1}|P_i|\left|\bar{P}_i\right|\left\|\Delta b_i\right\|^p\right)^{\frac{1}{p}} \\ \qquad\qquad\qquad\qquad\qquad \text{for } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\[2ex] \sum\limits_{i=1}^{n-1}|P_i|\left|\bar{P}_i\right|\left\|\dfrac{1}{\bar{P}_i}\bar{A}_i(\mathbf{p})-\dfrac{1}{P_i}A_i(\mathbf{p})\right\|\cdot\max\limits_{1\le i\le n-1}\left\|\Delta b_i\right\|. \end{cases}$$

In particular, if $p_i=\frac{1}{n}$, $i\in\{1,\dots,n\}$, *then we have*

$$(3.8)\qquad |T_n(\mathbf{a},\mathbf{b})|$$
$$\le \frac{1}{n^2}\times\begin{cases} \max\limits_{1\le i\le n-1}\left\|\dfrac{1}{n-i}\sum\limits_{k=i+1}^{n}a_k-\dfrac{1}{i}\sum\limits_{k=1}^{i}a_k\right\|\sum\limits_{i=1}^{n-1}i(n-i)\left\|\Delta b_i\right\|; \\[2ex] \left(\sum\limits_{i=1}^{n-1}i(n-i)\left\|\dfrac{1}{n-i}\sum\limits_{k=i+1}^{n}a_k-\dfrac{1}{i}\sum\limits_{k=1}^{i}a_k\right\|^q\right)^{\frac{1}{q}}\left(\sum\limits_{i=1}^{n-1}i(n-i)\left\|\Delta b_i\right\|^p\right)^{\frac{1}{p}} \\ \qquad\qquad\qquad\qquad\qquad \text{for } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\[2ex] \sum\limits_{i=1}^{n-1}i(n-i)\left\|\dfrac{1}{n-i}\sum\limits_{k=i+1}^{n}a_k-\dfrac{1}{i}\sum\limits_{k=1}^{i}a_k\right\|\cdot\max\limits_{1\le i\le n-1}\left\|\Delta b_i\right\|. \end{cases}$$

The inequalities in (3.7) and (3.8) are sharp in the above mentioned sense.

A different approach may be considered if one uses the representation in terms of double sums for the Čebyšev functional provided by Theorem 5.

The following result holds.

Theorem 9. *With the above assumptions of Theorem 6, we have the inequalities*

$$
\begin{aligned}
(3.9)\quad & \left|T_n\left(\mathbf{p};\mathbf{a},\mathbf{b}\right)\right| \\
& \leq \left|P_n\right| \times \begin{cases} \max\limits_{1\leq i,j\leq n-1}\left\{\left|P_{\min\{i,j\}}\right|,\left|\bar{P}_{\max\{i,j\}}\right|\right\}\sum\limits_{i=1}^{n-1}\left\|\Delta a_i\right\|\sum\limits_{i=1}^{n-1}\left\|\Delta b_i\right\|; \\ \\ \left(\sum\limits_{i=1}^{n-1}\sum\limits_{j=1}^{n-1}\left|P_{\min\{i,j\}}\right|^q\left|\bar{P}_{\max\{i,j\}}\right|^q\right)^{\frac{1}{q}}\left(\sum\limits_{i=1}^{n-1}\left\|\Delta a_i\right\|^p\right)^{\frac{1}{p}}\left(\sum\limits_{i=1}^{n-1}\left\|\Delta b_i\right\|^p\right)^{\frac{1}{p}} \\ \hfill \text{for } \ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \sum\limits_{i=1}^{n-1}\sum\limits_{j=1}^{n-1}\left|P_{\min\{i,j\}}\right|\left|\bar{P}_{\max\{i,j\}}\right|\max\limits_{1\leq i\leq n-1}\left\|\Delta a_i\right\|\max\limits_{1\leq i\leq n-1}\left\|\Delta b_i\right\|. \end{cases}
\end{aligned}
$$

The inequalities are sharp in the sense mentioned above.

The proof follows by the identity (2.7) on using Hölder's inequality for double sums and we omit the details.

Now, define

$$k_\infty := \max_{1\leq i,j\leq n-1}\left\{\frac{\min\{i,j\}}{n}\left(1-\frac{\max\{i,j\}}{n}\right)\right\},\quad n\geq 2.$$

Using the elementary inequality

$$ab\leq\frac{1}{4}\left(a+b\right)^2,\qquad a,b\in\mathbb{R};$$

we deduce

$$\min\{i,j\}\left(n-\max\{i,j\}\right)\leq\frac{1}{4}\left(n-|i-j|\right)^2$$

for $1\leq i,j\leq n-1$. Consequently, we have

$$k_\infty\leq\frac{1}{4n^2}\max_{1\leq i,j\leq n-1}\left\{\left(n-|i-j|\right)^2\right\}=\frac{1}{4}.$$

We may now state the following corollary of Theorem 9.

Corollary 3. *With the assumptions of Theorem 6 for* $\mathbf{a}$ *and* $\mathbf{b}$*, we have the inequality*

$$
\begin{aligned}
(3.10)\qquad \left|T_n\left(\mathbf{a},\mathbf{b}\right)\right| &\leq k_\infty\sum_{i=1}^{n-1}\left\|\Delta a_i\right\|\sum_{i=1}^{n-1}\left\|\Delta b_i\right\| \\
&\leq\frac{1}{4}\sum_{i=1}^{n-1}\left\|\Delta a_i\right\|\sum_{i=1}^{n-1}\left\|\Delta b_i\right\|.
\end{aligned}
$$

The constant $\frac{1}{4}$ *cannot be replaced in general by a smaller constant.*

Remark 2. *The inequality (3.10) is better than the third inequality in (1.11).*

Consider now, for $q>1$, the number

$$k_q := \frac{1}{n^2}\left(\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\left[\min\{i,j\}\left(n-\max\{i,j\}\right)\right]^q\right)^{\frac{1}{q}}.$$

We observe, by symmetry of the terms under the summation symbol, we have that

$$k_q = \frac{1}{n^2}\left(2\sum_{1\le i<j\le n-1} i^q (n-j)^q + \sum_{i=1}^{n-1} i^q (n-i)^q\right)^{\frac{1}{q}},$$

that may be computed exactly if $q = 2$ or another natural number.

Since, as above,

$$[\min\{i,j\}(n-\max\{i,j\})]^q \le \frac{1}{4^q}(n-|i-j|)^{2q},$$

we deduce

$$\begin{aligned} k_q &\le \frac{1}{4n^2}\left(\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}(n-|i-j|)^{2q}\right)^{\frac{1}{q}} \\ &\le \frac{1}{4n^2}\left[(n-1)^2 n^{2q}\right]^{\frac{1}{q}} \\ &= \frac{1}{4}(n-1)^{\frac{2}{q}}. \end{aligned}$$

Consequently, we may state the following corollary as well.

Corollary 4. *With the assumptions of Theorem 6 for* $\mathbf{a}$ *and* $\mathbf{b}$, *we have the inequalities*

$$\begin{aligned} |T_n(\mathbf{a},\mathbf{b})| &\le k_q\left(\sum_{i=1}^{n-1}\|\Delta a_i\|^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^{n-1}\|\Delta b_i\|^p\right)^{\frac{1}{p}} \\ &\le \frac{1}{4}(n-1)^{\frac{2}{q}}\left(\sum_{i=1}^{n-1}\|\Delta a_i\|^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^{n-1}\|\Delta b_i\|^p\right)^{\frac{1}{p}}, \end{aligned} \tag{3.11}$$

provided $p > 1$, $\frac{1}{p}+\frac{1}{q} = 1$. *The constant* $\frac{1}{4}$ *cannot be replaced in general by a smaller constant.*

Finally, if we denote

$$k_1 := \frac{1}{n^2}\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}[\min\{i,j\}(n-\max\{i,j\})],$$

then we observe, for $\mathbf{u} = \left(\frac{1}{n},\dots,\frac{1}{n}\right)$, $\mathbf{e} = (1,2,\dots,n)$, that

$$k_1 = T_n(\mathbf{u};\mathbf{e},\mathbf{e}) = \frac{1}{n}\sum_{i=1}^{n} i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} i\right)^2 = \frac{1}{12}\left(n^2-1\right),$$

and by Theorem 9, we deduce the inequality

$$|T_n(\mathbf{a},\mathbf{b})| \le \frac{1}{12}\left(n^2-1\right)\max_{1\le j\le n-1}\|\Delta a_j\|\max_{1\le j\le n-1}\|\Delta b_j\|.$$

Note that, the above inequality has been discovered using a different method in [3]. The constant $\frac{1}{12}$ is best possible.

4. Reverses for Jensen's Inequality

Let $(H;\langle\cdot,\cdot\rangle)$ be a real inner product space and $F:H\to\mathbb{R}$ a Fréchet differentiable convex function on H. If $\nabla F:H\to H$ denotes the gradient operator associated to F, then we have the inequality

$$F(x)-F(y)\geq\langle\nabla F(y),x-y\rangle$$

for each $x,y\in H$.

The following result has been obtained in [3].

Theorem 10. *Let $F:H\to\mathbb{R}$ be as above and $z_i\in H$, $i\in\{1,\dots,n\}$. If $q_i\geq 0$ $(i\in\{1,\dots,n\})$ with $\sum_{i=1}^{n}q_i=1$, then we have the following reverse of Jensen's inequality*

$$0\leq\sum_{i=1}^{n}q_iF(z_i)-F\left(\sum_{i=1}^{n}q_iz_i\right) \tag{4.1}$$
$$\leq\begin{cases}\left[\sum\limits_{i=1}^{n}i^2q_i-\left(\sum\limits_{i=1}^{n}iq_i\right)^2\right]\max\limits_{k=1,\dots,n-1}\left\|\Delta\left(\nabla F(z_i)\right)\right\|\max\limits_{k=1,\dots,n-1}\left\|\Delta z_i\right\|;\\[2ex] \left[\sum\limits_{1\leq j<i\leq n}q_iq_j(i-j)\right]\left(\sum\limits_{i=1}^{n-1}\left\|\Delta\left(\nabla F(z_i)\right)\right\|^p\right)^{\frac{1}{p}}\left(\sum\limits_{i=1}^{n-1}\left\|\Delta z_i\right\|^q\right)^{\frac{1}{q}}\\ \hfill \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2ex] \frac{1}{2}\left[\sum\limits_{i=1}^{n}q_i(1-q_i)\right]\sum\limits_{i=1}^{n-1}\left\|\Delta\left(\nabla F(z_i)\right)\right\|\sum\limits_{i=1}^{n-1}\left\|\Delta z_i\right\|.\end{cases}$$

The unweighted case may useful in application and is incorporated in the following corollary.

Corollary 5. *Let $F:H\to\mathbb{R}$ be as above and $z_i\in H$, $i\in\{1,\dots,n\}$. Then we have the inequalities*

$$0\leq\frac{1}{n}\sum_{i=1}^{n}F(z_i)-F\left(\frac{1}{n}\sum_{i=1}^{n}z_i\right)$$
$$\leq\begin{cases}\dfrac{n^2-1}{12}\max\limits_{k=1,\dots,n-1}\left\|\Delta\left(\nabla F(z_k)\right)\right\|\max\limits_{k=1,\dots,n-1}\left\|\Delta z_k\right\|;\\[2ex] \dfrac{n^2-1}{6n}\left(\sum\limits_{k=1}^{n-1}\left\|\Delta\left(\nabla F(z_k)\right)\right\|^p\right)^{\frac{1}{p}}\left(\sum\limits_{k=1}^{n-1}\left\|\Delta z_k\right\|^q\right)^{\frac{1}{q}}\\ \hfill \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2ex] \dfrac{n-1}{2n}\sum\limits_{k=1}^{n-1}\left\|\Delta\left(\nabla F(z_k)\right)\right\|\sum\limits_{k=1}^{n-1}\left\|\Delta z_k\right\|.\end{cases}$$

By making use, of Theorem 9, we can state the following result as well:

Theorem 11. *Let $F : H \to \mathbb{R}$ be as above and $z_i \in H$, $i \in \{1, \ldots, n\}$. If $q_i \geq 0$ $(i \in \{1, \ldots, n\})$ with $\sum_{i=1}^{n} q_i = 1$, then we have the following reverse of Jensen's inequality*

$$0 \leq \sum_{i=1}^{n} q_i F(z_i) - F\left(\sum_{i=1}^{n} q_i z_i\right) \tag{4.2}$$

$$\leq \begin{cases} \max_{1 \leq i,j \leq n-1} \left\{Q_{\min\{i,j\}}, \overline{Q}_{\max\{i,j\}}\right\} \sum_{i=1}^{n-1} \|\Delta(\nabla F(z_i))\| \sum_{i=1}^{n-1} \|\Delta z_i\|; \\ \left(\sum_{i=1}^{n-1} \sum_{j=1}^{n-1} Q_{\min\{i,j\}}^{q} \overline{Q}_{\max\{i,j\}}^{q}\right)^{\frac{1}{q}} \left(\sum_{i=1}^{n-1} \|\Delta(\nabla F(z_i))\|^{p}\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n-1} \|\Delta z_i\|^{p}\right)^{\frac{1}{p}} \\ \qquad \textit{for } \ p > 1, \ \frac{1}{p} + \frac{1}{q} = 1; \\ \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} Q_{\min\{i,j\}} \overline{Q}_{\max\{i,j\}} \max_{1 \leq i \leq n-1} \|\Delta(\nabla F(z_i))\| \max_{1 \leq i \leq n-1} \|\Delta z_i\|. \end{cases}$$

Proof. We know, see for example [1, Eq. (4.4)], that the following reverse of Jensen's inequality for Fréchet differentiable convex functions

$$\begin{aligned} 0 &\leq \sum_{i=1}^{n} q_i F(z_i) - F\left(\sum_{i=1}^{n} q_i z_i\right) \\ &\leq \sum_{i=1}^{n} q_i \langle \nabla F(z_i), z_i \rangle - \left\langle \sum_{i=1}^{n} q_i \nabla F(z_i), \sum_{i=1}^{n} q_i z_i \right\rangle \end{aligned} \tag{4.3}$$

holds.

Now, if we apply Theorem 9 for the choices $a_i = \nabla F(z_i), b_i = z_i$ and $p_i = q_i \ (i = 1, ..., n)$, then we may state

$$\sum_{i=1}^{n} q_i \langle \nabla F(z_i), z_i \rangle - \left\langle \sum_{i=1}^{n} q_i \nabla F(z_i), \sum_{i=1}^{n} q_i z_i \right\rangle \tag{4.4}$$

$$\leq \begin{cases} \max_{1 \leq i,j \leq n-1} \left\{Q_{\min\{i,j\}}, \overline{Q}_{\max\{i,j\}}\right\} \sum_{i=1}^{n-1} \|\Delta(\nabla F(z_i))\| \sum_{i=1}^{n-1} \|\Delta z_i\|; \\ \left(\sum_{i=1}^{n-1} \sum_{j=1}^{n-1} Q_{\min\{i,j\}}^{q} \overline{Q}_{\max\{i,j\}}^{q}\right)^{\frac{1}{q}} \left(\sum_{i=1}^{n-1} \|\Delta(\nabla F(z_i))\|^{p}\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n-1} \|\Delta z_i\|^{p}\right)^{\frac{1}{p}} \\ \text{for } \ p > 1, \ \frac{1}{p} + \frac{1}{q} = 1; \\ \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} Q_{\min\{i,j\}} \overline{Q}_{\max\{i,j\}} \max_{1 \leq i \leq n-1} \|\Delta(\nabla F(z_i))\| \max_{1 \leq i \leq n-1} \|\Delta z_i\|. \end{cases}$$

Finally, on making use of the inequalities (4.3) and (4.4), we deduce the desired result (4.2). ∎

The unweighted case may be useful in application and is incorporated in the following corollary.

Corollary 6. *Let $F : H \to \mathbb{R}$ be as above and $z_i \in H$, $i \in \{1, \dots, n\}$. Then we have the inequalities*

$$0 \leq \frac{1}{n}\sum_{i=1}^{n} F(z_i) - F\left(\frac{1}{n}\sum_{i=1}^{n} z_i\right)$$

$$\leq \begin{cases} \dfrac{n^2-1}{12} \max\limits_{k=1,\dots,n-1} \left\|\Delta\left(\nabla F(z_k)\right)\right\| \max\limits_{k=1,\dots,n-1} \left\|\Delta z_k\right\|; \\ \\ \frac{1}{4}(n-1)^{\frac{2}{q}} \left(\sum\limits_{k=1}^{n-1} \left\|\Delta\left(\nabla F(z_k)\right)\right\|^p\right)^{\frac{1}{p}} \left(\sum\limits_{k=1}^{n-1} \left\|\Delta z_k\right\|^p\right)^{\frac{1}{p}} \\ \hfill \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \frac{1}{4}\sum\limits_{k=1}^{n-1} \left\|\Delta\left(\nabla F(z_k)\right)\right\| \sum\limits_{k=1}^{n-1} \left\|\Delta z_k\right\|. \end{cases}$$

Remark 3. *If one applies the other Grüss' type inequalities obtained in the previous section, then one can obtain other reverses for Jensen's discrete inequality for convex functions defined on inner product spaces. We do not present them here.*

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